Control Systems I

Frequency Response

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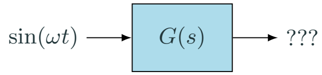
System Frequency Response

Describe the behaviour of the system by how it responds to sinusoidal inputs



Key take-home:

- The output is a sinusoid of a same frequency with a phase shift of $\angle G(j\omega)$ and a magnitude of $|G(j\omega)|$

Why important?

- Very useful method to experimentally capture the dynamics of a system
- Common control objectives expressed in terms of frequency requirements
- Can determine closed-loop behaviour, from open-loop frequency response

The Math

Suppose we have a system with a transfer function $G(s)$, and we drive it with the input signal $u(t) = \sin(\omega t)$.

To make things simple, drive the system with the complex signal:

$$u(t) := e^{j\omega t}$$

[1]Assume for now that G only has simple poles that are all different from $j\omega$

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$$u(t) := e^{j\omega t}$$

The output is

$$\begin{aligned} Y(s) &= G(s)U(s) \\ &= G(s)\frac{1}{s - j\omega} \\ &= c_0 + \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \cdots + \frac{c_n}{s - p_n} + \frac{c}{s - j\omega} \end{aligned}$$

where $\{p_i\}$ are the unique, simple poles of $G(s)$[1]

[1]Assume for now that G only has simple poles that are all different from $j\omega$

Computing the Steady-State Response

The response to $e^{j\omega t}$ is

$$y(t) = c_0\delta(t) + c_1 e^{p_1 t} + \cdots + c_n e^{p_n t} + c e^{j\omega t}$$

If $G(s)$ is stable, then for sufficiently large t, this will tend to

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Compute c in the standard fashion:

$$\begin{aligned} c &= \lim_{s \to j\omega} (s - j\omega) G(s) U(s) \\ &= \lim_{s \to j\omega} (s - j\omega) G(s) \frac{1}{s - j\omega} \\ &= G(j\omega) \end{aligned}$$

Steady-State Response

Use superposition to get the response to $u(t) = \sin(\omega t)$

$$u(t) = \sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

For large t we have that

$$\begin{aligned} y(t) &= \frac{1}{2j}\left(G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}\right) \\ &= \frac{|G(j\omega)|}{2j}\left(e^{j\angle G(j\omega)}e^{j\omega t} - e^{-j\angle G(j\omega)}e^{-j\omega t}\right) \\ &= \frac{|G(j\omega)|}{2j}\left(e^{j(\angle G(j\omega)+\omega t)} - e^{-j(\angle G(j\omega)+\omega t)}\right) \\ &= |G(j\omega)| \sin(\omega t + \angle G(j\omega)) \end{aligned}$$

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If the input is a sinusoid at frequency ω, the output is a sinusoid at the same frequency, with the magnitude scaled by $|G(j\omega)|$ and the phase shifted by $\angle G(j\omega)$.

Example

Consider the system $\ddot{y}(t) + 1.1\dot{y}(t) + 0.1y(t) = u(t)$ diven by $u(t) = \sin(1.3t)$

[1] Recall that $B\cos(\omega_0 t) + C\sin(\omega_0 t) = A\sin(\omega_0 t + \phi)$, where $\phi = \tan^{-1}\left(\frac{C}{B}\right)$ and $A = \sqrt{B^2 + C^2}$

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The output is

$$\begin{aligned} Y(s) = G(s)U(s) &= \frac{1}{(s+1)(s+0.1)} \cdot \frac{1.3}{s^2 + 1.3^2} \\ &= \frac{-0.54}{s+1} + \frac{0.85}{s+0.1} - \frac{0.31s + 0.45}{s^2 + 1.3^2} \end{aligned}$$

Taking the inverse transform we get the time response

$$\begin{aligned} y(t) &= -0.54e^{-t} + 0.85e^{-0.1t} - 0.31\cos(1.3t) - 0.45\sin(1.3t) \\ &= -0.54e^{-t} + 0.85e^{-0.1t} + 0.47\sin(1.3t - 2.41) \end{aligned}$$

Note that

$$G(j1.3) = -0.35 - 0.31j = 0.47e^{-2.41j}$$

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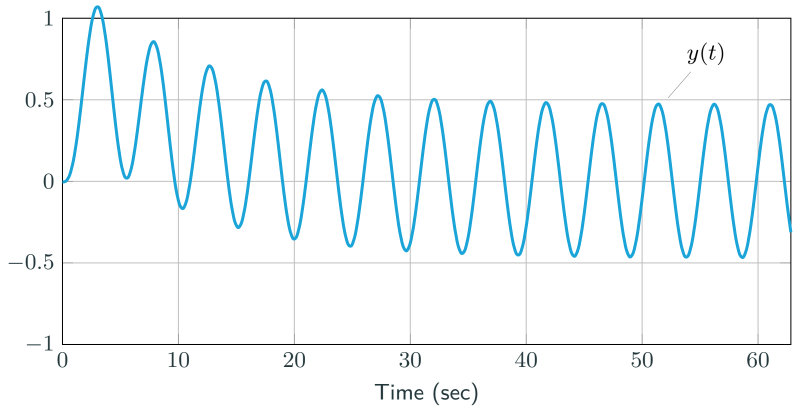
Example

$$y(t) = -0.54e^{-t} + 0.85e^{-0.1t} + 0.47\sin(1.3t - 2.41)$$



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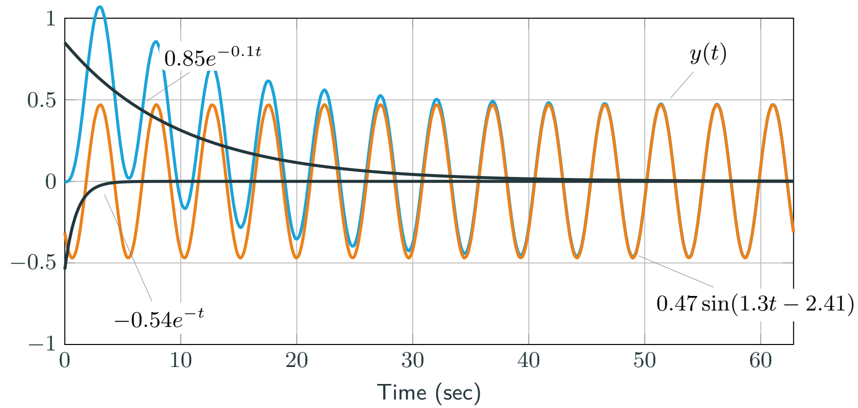

Only the sinusoid counts in the long-term.
Initial response is called the ***transient reponse***, the long-term is called the ***steady-state*** response

Frequency Response

The ***frequency response*** of the transfer function $G(s)$ is the function $G(j\omega)$

- If any of the poles are unstable, $\operatorname{Re} p_i > 0$, then the transient will not die-out

$$\lim_{t\to\infty} |e^{-p_i t}| = \infty$$

- If any poles are on the imaginary axis, $p_i = j\omega_i$, then their response will not die out either, but will tend to a sinusoid with a frequency of ω_i.

In these cases, the frequency response is still well-defined, ***but it no longer defines the steady-state response!***

Complex Sinusoidal Inputs

Phase shifted input

$$u(t) = \sin(\omega t + \phi)$$

What is $y(t)$?

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Change of variables $\tau = t + \frac{\phi}{\omega}$

$$u(\tau) = \sin(\omega\tau)$$
$$\Rightarrow y(\tau) = |G(j\omega)| \sin(\omega\tau + \angle G(j\omega))$$

Undoing change of variables gives

$$y(t) = |G(j\omega)| \sin(\omega t + \phi + \angle G(j\omega))$$

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Idea: Which time is 'zero' doesn't matter when we're talking about a signal running infinitely far into the past and into the future

Example

$$u(t) = \sin(2\pi/3t) + 1.2\sin(\pi t + 0.3) + 0.8\sin(2\pi/5t + 0.4)$$

What's the steady-output of a system with transfer function $G(s)$, if the system has all poles in the left half plane?

Example

$$u(t) = \sin(2\pi/3t) + 1.2\sin(\pi t + 0.3) + 0.8\sin(2\pi/5t + 0.4)$$

What's the steady-output of a system with transfer function $G(s)$, if the system has all poles in the left half plane?

Superposition gives:

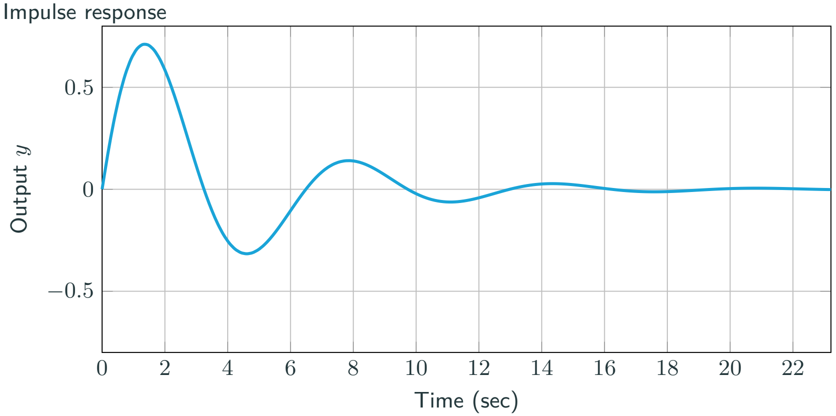
$$\begin{aligned} y(t) = &|G(j2\pi/3)|\sin(2\pi/3t + \angle G(j2\pi/3)) \\ &+ 1.2\,|G(j\pi)|\sin(\pi t + 0.3 + \angle G(j\pi)) \\ &+ 0.8\,|G(j2\pi/5)|\sin(2\pi/5t + 0.4 + \angle G(j2\pi/5)) \end{aligned}$$

Example

Consider the simple system

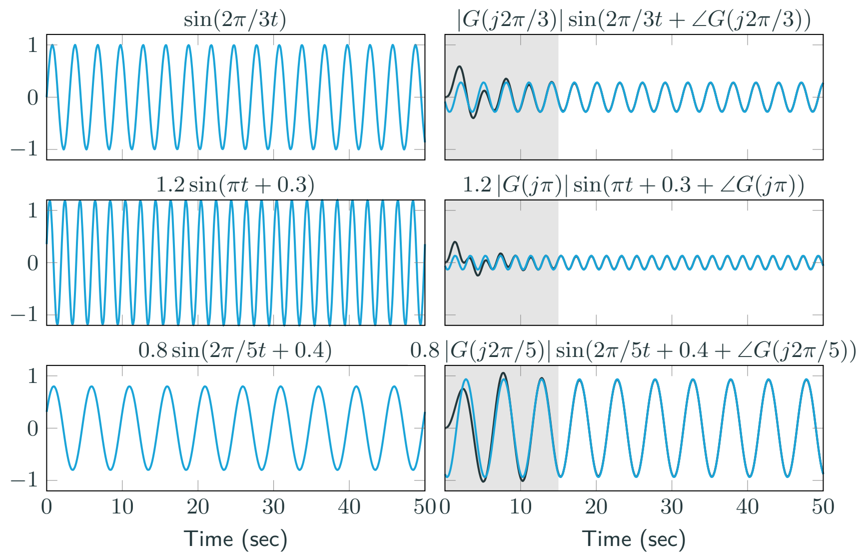
$$G(s) = \frac{1}{s^2 + 0.5s + 1}$$

Impulse response

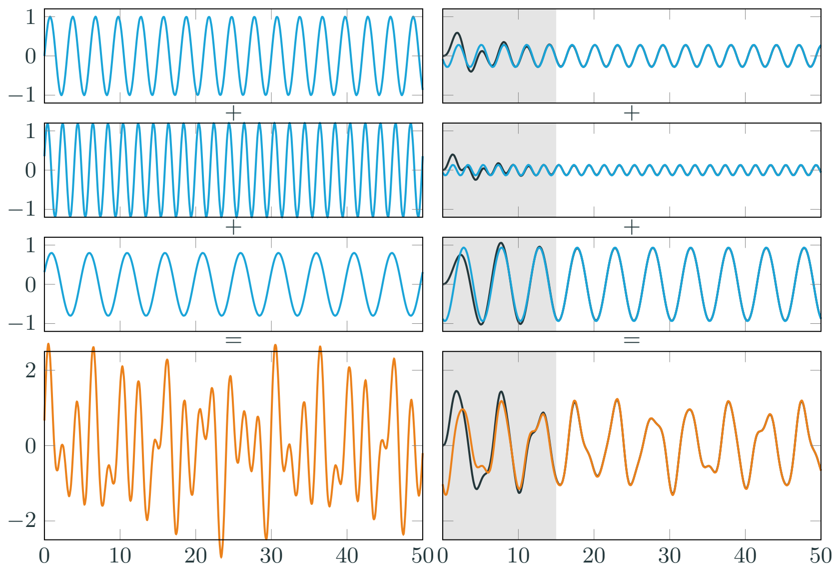


Expect the transient phase to die out around 15 seconds.

Example

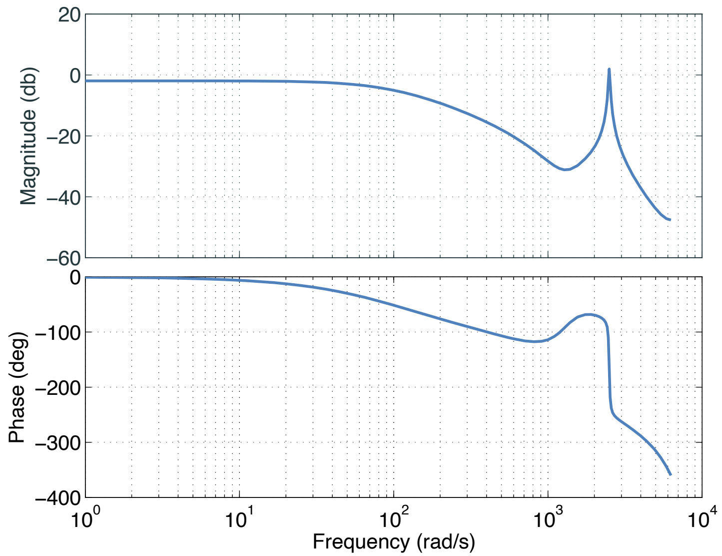


Example



Visualizing the Frequency Response

Visualization: Bode Plot



Bode Plots

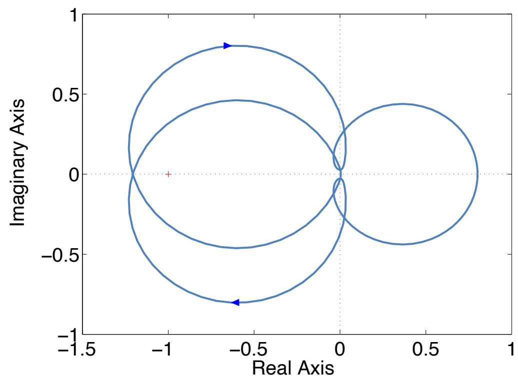
1. Magnitude plot $|G(j\omega)|$
   - Plotted in decibels $20\log_{10}(|G(j\omega)|)$
   - X-axis is frequency, usually rad/sec, but sometimes Hz
   - Value above 0 → the output is larger than the input
   - Value below 0 → the output is smaller than the input
   - All physical systems will tend to $-\infty$ decibels as $\omega \to \infty$

2. Phase plot $\angle G(j\omega)$
   - Generally shown in degrees
   - X-axis is frequency, usually rad/sec, but sometimes Hz
   - Value above 0 → phase advance
   - Value above 0 → phase lag

Bode Plots

- Easy to generate directly from data
- Much can be said from a 'glance'
  - Gain at specific frequencies obvious
  - Resonance frequencies
  - Bandwidth
  - Stability in closed-loop
  - etc
- Commonly used for control design
- Control objectives commonly described using Bode plot
- Can be generalized to multi-input / multi-output systems
- 'Easy' to sketch manually

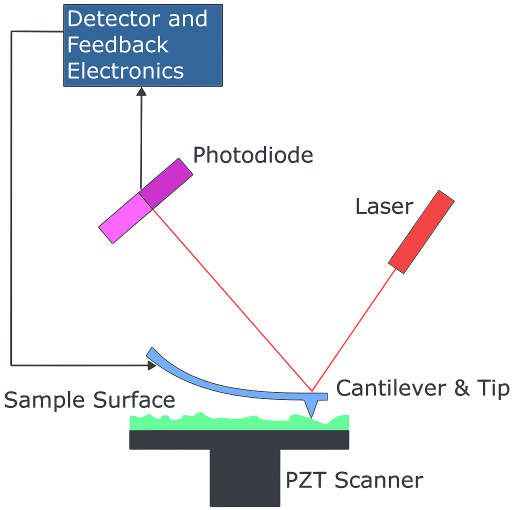
Nyquist Diagram



- Generally used for more theoretical analysis
  - e.g., robust stability, robust performance, etc
- Significantly more complex to draw
  - We could spend weeks learning this...
  - I'll just say "Use a computer" - `nyquist(G)`

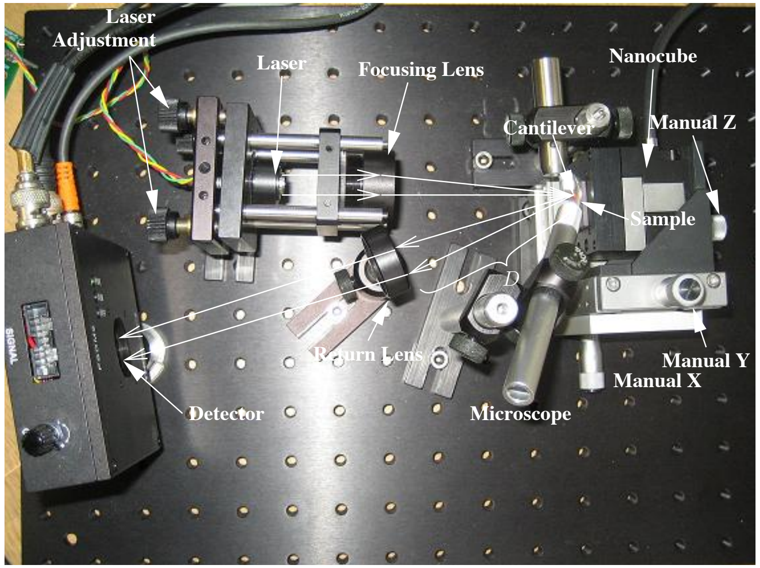
Example: Atomic Force Microscope

Atomic Force Microscope



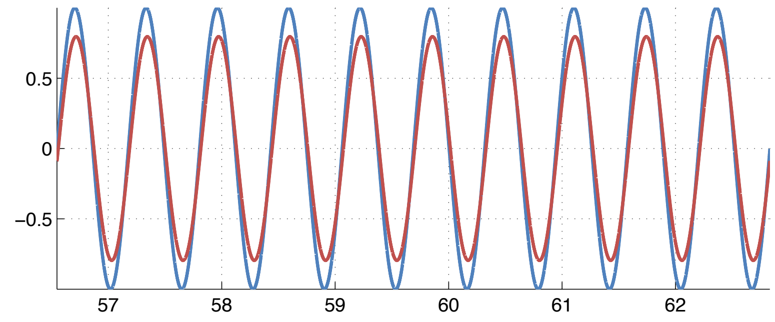
[1] Image: Wikipedia

Atomic Force Microscope: Student Experiment



Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i's



$\omega = 2\pi 10$ $|G(j\omega)| = 7.96 \cdot 10^{-1}$ $\angle G(j\omega) = -3.03$

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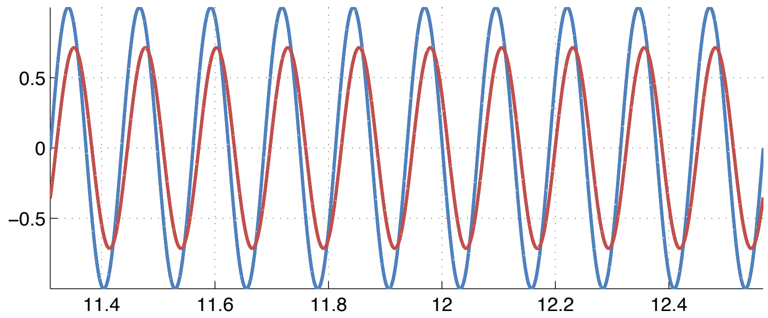
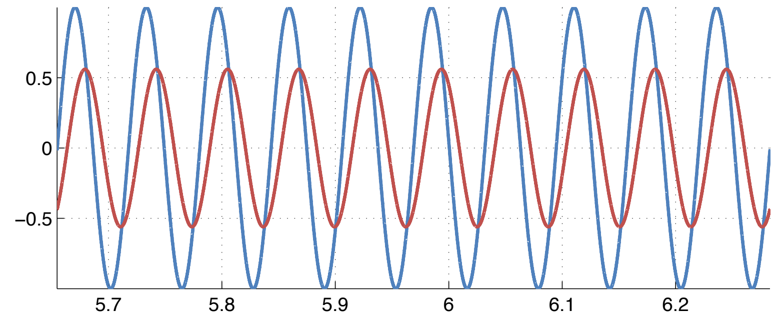


$\omega = 2\pi 50$ $|G(j\omega)| = 7.14 \cdot 10^{-1}$ $\angle G(j\omega) = -2.63$

Generating a Bode Plot: Method 1

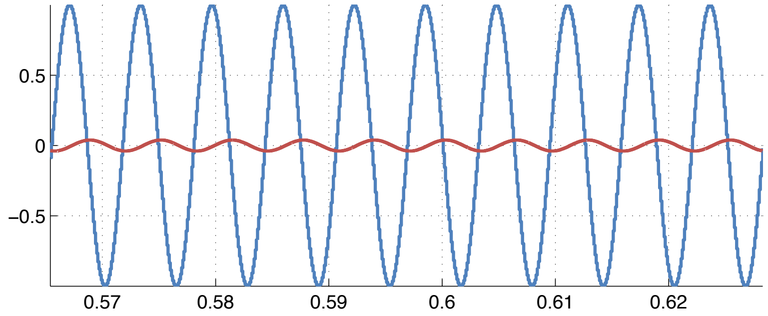
Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i's



$\omega = 2\pi 100$ $|G(j\omega)| = 5.61 \cdot 10^{-1}$ $\angle G(j\omega) = -2.26$

Generating a Bode Plot: Method 1

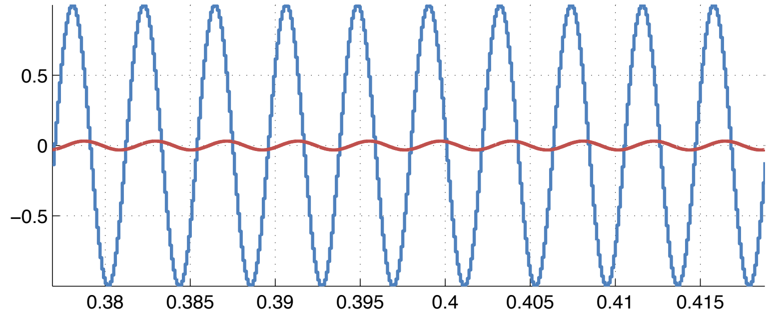
Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i's



$\omega = 2\pi 1000$ $\quad |G(j\omega)| = 3.92 \cdot 10^{-2}$ $\quad \angle G(j\omega) = -1.35$

Generating a Bode Plot: Method 1

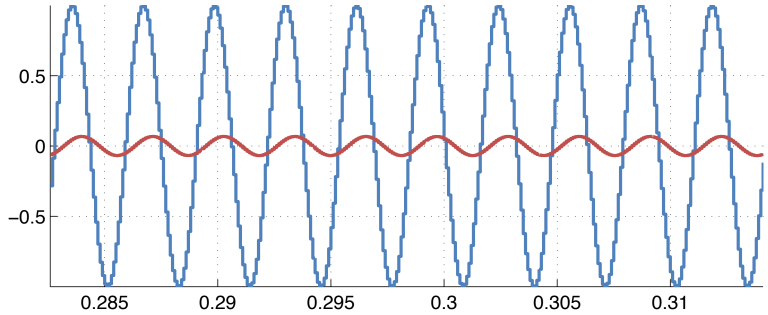
Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i's



$\omega = 2\pi 1500$ $\quad |G(j\omega)| = 3.16 \cdot 10^{-2}$ $\quad \angle G(j\omega) = -2.11$

Generating a Bode Plot: Method 1

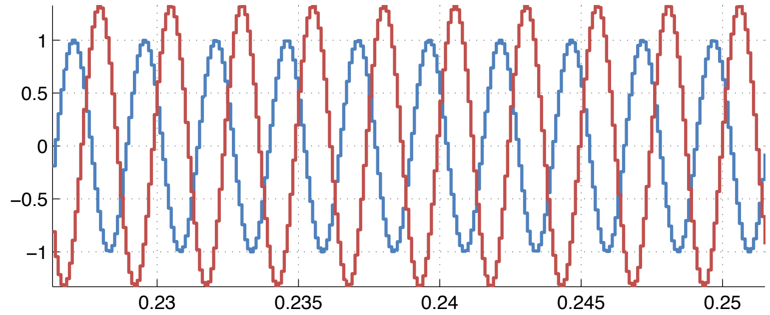
Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i's



$\omega = 2\pi 2000$ $\quad |G(j\omega)| = 6.80 \cdot 10^{-2}$ $\quad \angle G(j\omega) = -2.33$

Generating a Bode Plot: Method 1

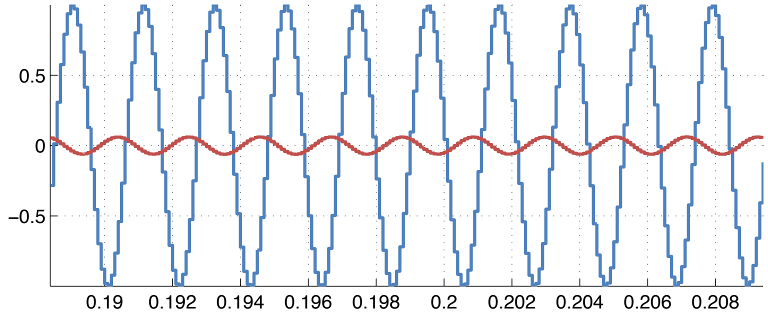
Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i's



$\omega = 2\pi 2498$ $\quad |G(j\omega)| = 1.33$ $\quad \angle G(j\omega) = -0.85$

Generating a Bode Plot: Method 1

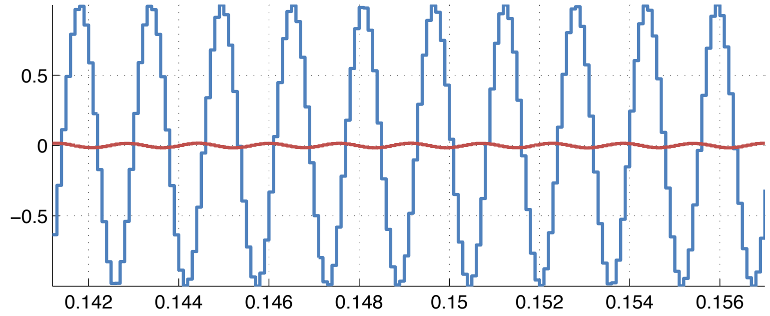
Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i's



$\omega = 2\pi 3000$ $\quad |G(j\omega)| = 6.16 \cdot 10^{-2}$ $\quad \angle G(j\omega) = -0.84$

Generating a Bode Plot: Method 1

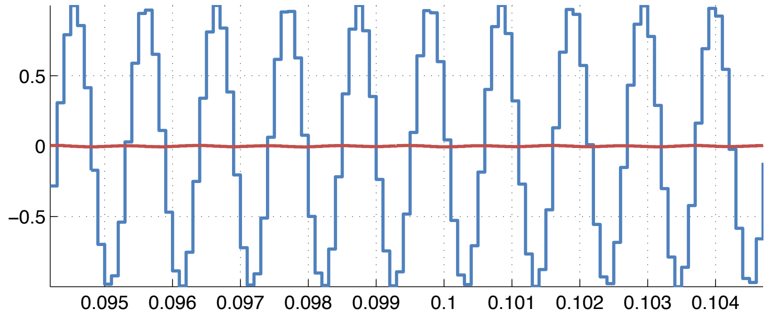
Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i's



$\omega = 2\pi 4000$ $\quad |G(j\omega)| = 1.67 \cdot 10^{-2}$ $\quad \angle G(j\omega) = -1.07$

Generating a Bode Plot: Method 1

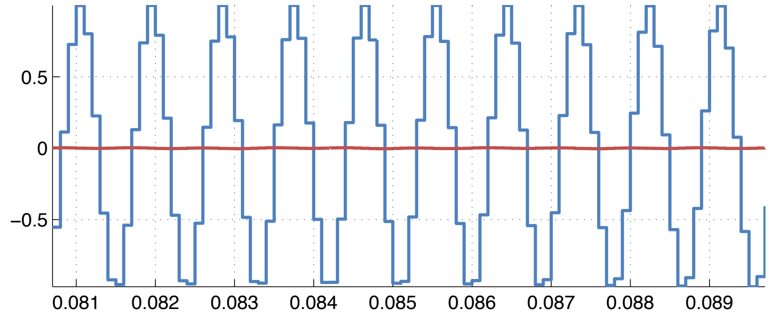
Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i's



$\omega = 2\pi 6000$ $\quad |G(j\omega)| = 5.25 \cdot 10^{-3}$ $\quad \angle G(j\omega) = -1.40$

Generating a Bode Plot: Method 1

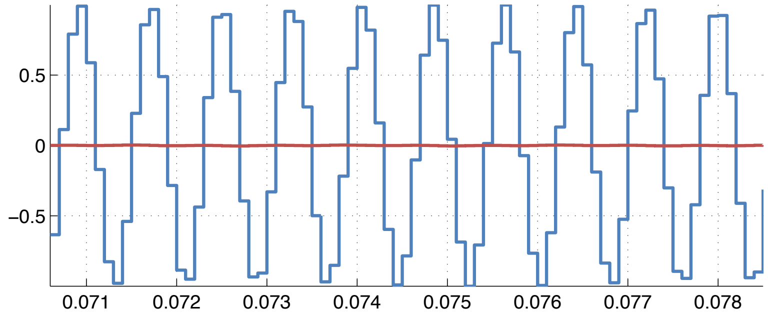
Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i's



$\omega = 2\pi 7000$ $\quad |G(j\omega)| = 3.54 \cdot 10^{-3}$ $\quad \angle G(j\omega) = -1.50$

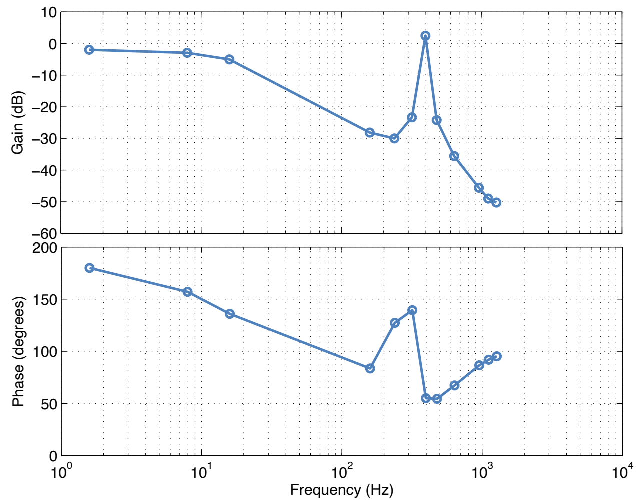
Generating a Bode Plot: Method 1

Drive the system with the input $u(t) = \sin(\omega_i t)$ for several ω_i's



$\omega = 2\pi 8000$ $\quad |G(j\omega)| = 3.06 \cdot 10^{-3}$ $\quad \angle G(j\omega) = -1.55$

Put Sampled Points on Plot

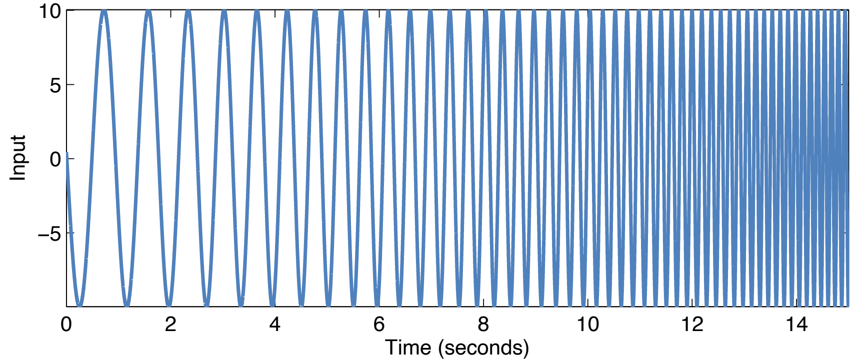


Problem: Need a *lot* of points to get a good fit

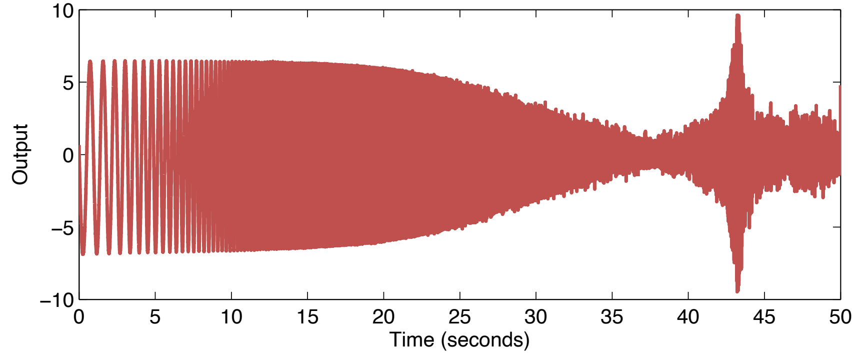
Better: Frequency Sweep

Sweep from f_0Hz to f_fHz in t_s seconds

$$u(t) = cos(\phi(t)) \qquad \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = f_0 \left(\frac{f_0}{f_f}\right)^{t/t_s}$$



Output in Response to the Frequency Sweep

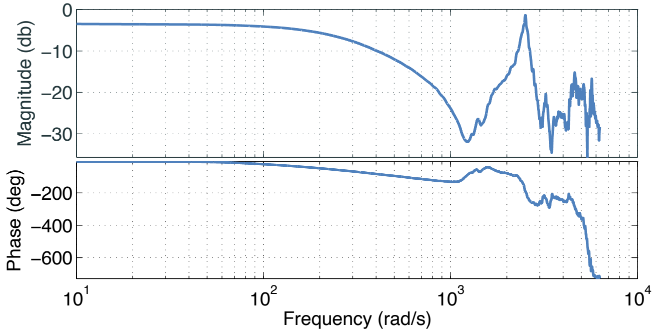


Obtaining the Frequency Response Experimentally

Recall the definition of frequency response:

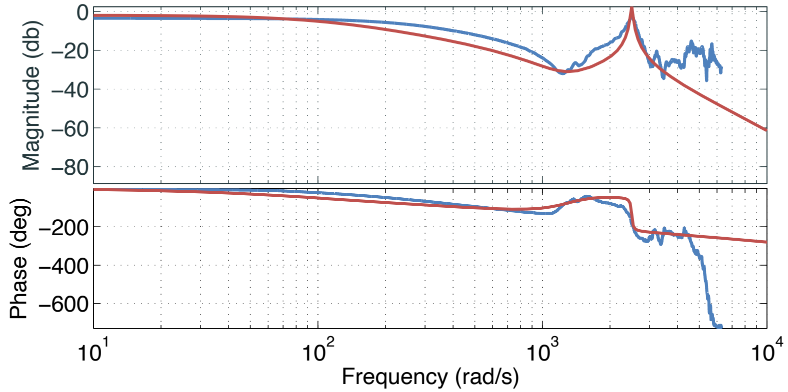
$$G(j\omega) = \frac{Y(j\omega)}{U(j\omega)}$$

and note that $Y(j\omega)$ is the discrete-time Fourier transform of $y(k)$. We just compute the Fourier transform of $y(k)$ and $u(k)$, and take their difference



Interpreting the Frequency Response

$$G(s) = \frac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s+3000)(s+1000)(s+100)(s^2+50s+6.25 \cdot 10^6)}$$

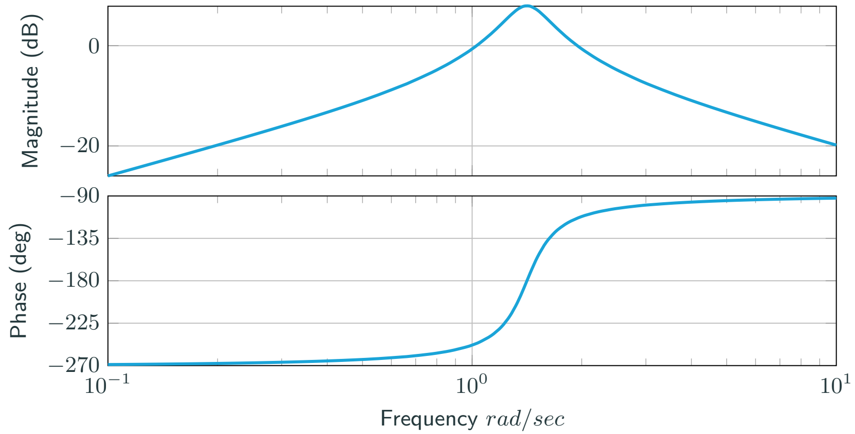


Recall: Sketching Bode Plots

Matlab Commands

The computer way: Grid ω, compute and plot.

```
sys = tf([1 0],[1 -0.4 2]); % Define system
bode(sys); % Show bode plot
```



Sketching Ratio of Polynomials on a Log-Scale

Sketch Bode plot of continuous-time system $G(s)$

$$G(s) = c\frac{(s+z_1)(s+z_2)}{s(s+p_1)(s+p_2)}$$

Goal: Sketch $G(j\omega)$

For any ω we have a complex number in the form

$$G(j\omega) = \frac{s_1 s_2}{s_3 s_4 s_5} = \frac{r_1 e^{j\theta_1} r_2 e^{j\theta_2}}{r_3 e^{j\theta_3} r_4 e^{j\theta_4} r_5 e^{j\theta_5}} = \frac{r_1 r_2}{r_3 r_4 r_5} e^{j(\theta_1+\theta_2-\theta_3-\theta_4-\theta_5)}$$

So the magnitude is linear in a log-scale, and the phase is linear

$$\begin{aligned} 20\log_{10}|G(j\omega)| &= 20\log_{10} r_1 + 20\log_{10} r_2 - 20\log_{10} r_3 - 20\log_{10} r_4 - 20\log_{10} r_5 \\ \angle G(j\omega) &= \theta_1 + \theta_2 - \theta_3 - \theta_4 - \theta_5 \end{aligned}$$

Write in Bode Form

Convert to Bode form

$$G(j\omega) = c_0 (j\omega)^n \frac{(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)\cdots}{(j\omega\tau_a + 1)(j\omega\tau_b + 1)\cdots}$$

Example:

$$G(s) = 15s^{-1}\frac{(\tau_1 s + 1)(\tau_2 s + 1)}{(\tau_3 s + 1)(\tau_4 s + 1)}$$

The magnitude then of the form

$$\begin{aligned} 20\log_{10}|G(j\omega)| = 20\log_{10} 15 + 20\log_{10}|j\omega\tau_1 + 1| + 20\log_{10}|j\omega\tau_2 + 1| \\ - 20\log_{10}|j\omega\tau_3 + 1| - 20\log_{10}|j\omega\tau_4 + 1| \end{aligned}$$

Standard Elements

All transfer functions are made of three types of terms

1. $K_0(j\omega)^n$
2. $(j\omega\tau + 1)^{\pm 1}$
3. $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{\pm 1}$

Integrator / Pole at Zero $K_0(j\omega)^n$

Magnitude

$$20\log_{10}|K_0(j\omega)^n| = 20\log_{10}|K_0| + n20\log_{10}|j\omega|$$

This is a straight line with slope of $n \times$ (20 decibels per decade).

Phase

$$\angle K_0(j\omega)^n = n \times 90^\circ$$

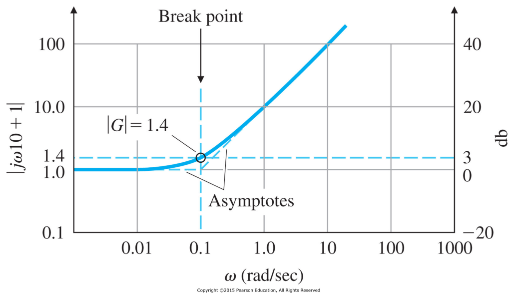
The phase is constant everywhere. Each integrator drops the phase by 90°, and each zero at zero increases it by 90°

Simple pole / zero $(j\omega\tau + 1)^{\pm 1}$

Magnitude $20 \log_{10} |j\omega\tau + 1|$

- $\omega\tau \ll 1 \quad \rightarrow \quad j\omega\tau + 1 \approxeq 1$
- $\omega\tau \gg 1 \quad \rightarrow \quad j\omega\tau + 1 \approxeq j\omega\tau$

Example: $j\omega 10 + 1$

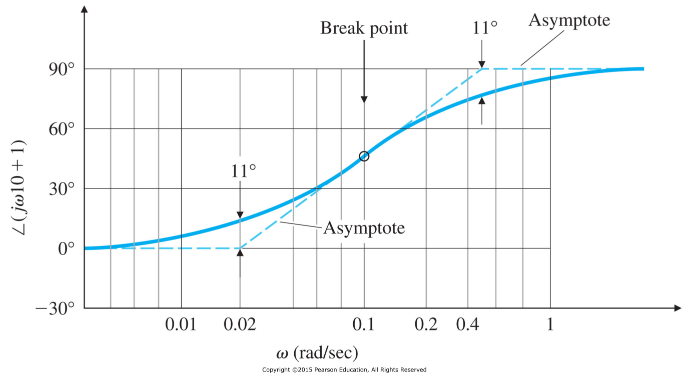


Simple pole / zero $(j\omega\tau + 1)^{\pm 1}$

Phase $\angle(j\omega\tau + 1)$

- $\omega\tau \ll 1 \quad \rightarrow \quad \angle 1 = 0^\circ$
- $\omega\tau \approxeq 1 \quad \rightarrow \quad \angle(j\omega\tau + 1) \approxeq 45^\circ$
- $\omega\tau \gg 1 \quad \rightarrow \quad \angle j\omega\tau = 90^\circ$

Example: $j\omega 10 + 1$



Second-order term $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{\pm 1}$

Magnitude

Very similar to first-order term, with breakpoint at $\omega = \omega_n$

- $\frac{\omega}{\omega_n} \ll 1 \quad \rightarrow \quad \left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1 \approxeq 1$
- $\frac{\omega}{\omega_n} \gg 1 \quad \rightarrow \quad \left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1 \approxeq \left(\frac{j\omega}{\omega_n}\right)^2$

Second-order term $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{\pm 1}$

Magnitude

Very similar to first-order term, with breakpoint at $\omega = \omega_n$

- $\frac{\omega}{\omega_n} \ll 1 \quad \rightarrow \quad \left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1 \approxeq 1$
- $\frac{\omega}{\omega_n} \gg 1 \quad \rightarrow \quad \left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1 \approxeq \left(\frac{j\omega}{\omega_n}\right)^2$

The magnitude around the crossover frequency is impacted by the damping ratio

$$\begin{aligned} |G(j\omega_n)| &= \left|\left(\frac{j\omega_n}{\omega_n}\right)^2 + 2\zeta\frac{j\omega_n}{\omega_n} + 1\right|^n \\ &= |-1 + 2\zeta j + 1|^n \\ &= |2\zeta|^n \end{aligned}$$

So for the most common case of $n = -1$ (a resonant pole) we have

$$|G(j\omega_n)| \approxeq \frac{1}{2\zeta}$$

Second order terms

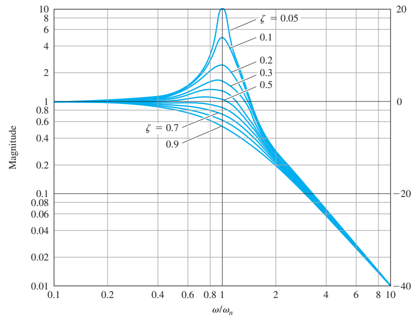
Phase

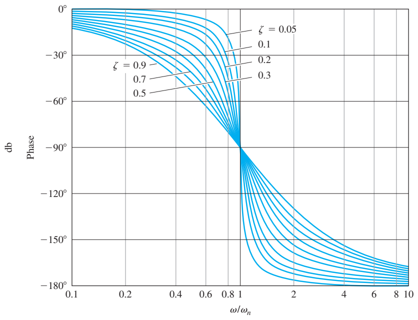
$$G(j\omega) = \left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$$

- $\omega \ll \omega_n \quad \rightarrow \quad \angle 1 = 0^\circ$
- $\omega \approxeq \omega_n \quad \rightarrow \quad \angle j^2 + 2\zeta j + 1 = 90^\circ$
- $\omega \gg \omega_n \quad \rightarrow \quad \angle\left(\frac{j\omega}{\omega_n}\right)^2 = 180^\circ$

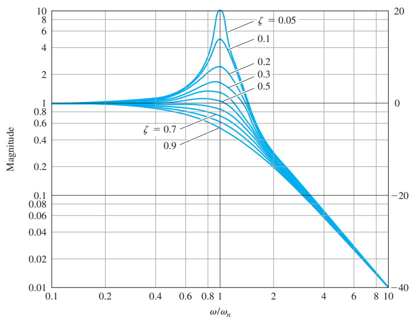
Second-Order Poles / Zeros

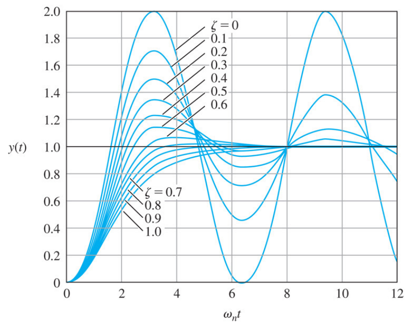
$$G(j\omega) = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$





Second-Order Poles / Zeros $\quad G(w) = \frac{1}{(w/\omega_n)^2+2\zeta(w/\omega_n)+1}$





- Damping ζ
  - $\rightarrow$ transient-response overshoot (approx $1/2\zeta$ for $\zeta < 0.5$)
  - $\rightarrow$ peak in frequency response magnitude
- Natural frequency ω_n
  - $\rightarrow$ approximately equal to bandwidth
  - $\rightarrow$ proportional to the rise time

Unstable Poles and Non-minimum Phase Zeros

What happens if the pole/zero is in the right half plane?

$$G_1(j\omega) = j\omega\tau + 1 \qquad\qquad G_2(j\omega) = j\omega\tau - 1$$

Magnitude

$$|G_1(j\omega)| = |G_2(j\omega)|$$

Magnitude is the same

Phase

- $\omega\tau \ll 1 \quad \rightarrow \quad \angle G_1(j\omega) = 0^\circ \quad \angle G_2(j\omega) = 180^\circ$
- $\omega\tau \approxeq 1 \quad \rightarrow \quad \angle G_1(j\omega) \approxeq 45^\circ \quad \angle G_2(j\omega) \approxeq 135^\circ$
- $\omega\tau \gg 1 \quad \rightarrow \quad \angle G_1(j\omega) = 90^\circ \quad \angle G_2(j\omega) = 90^\circ$

Phase goes in the opposite direction for RHP poles / zeros

Understanding Bode Plots

Magnitude

- Each zero increases the slope by 20dB/dec
- Each pole decreases the slope by 20dB/dec
- Complex poles/zeros have a resonant peak; larger with lower damping ratio
- Physical systems must have a negative slope as $\omega \to \infty$
- Slope changes occur at pole/zero locations

Phase

- Negative zero $\to 90^\circ$, positive zero $\to -90^\circ$
- Negative pole $\to -90^\circ$, positive pole $\to 90^\circ$
- Physical systems must have a negative phase as $\omega \to \infty$
- Phase changes begin/end $\sim 1/2$ decade before/after poles/zeros

Three Indicative Examples

1. $G(s) = \dfrac{2000(s+0.5)}{s(s+10)(s+50)}$
2. $G(s) = \dfrac{10}{s(s^2+0.4s+4)}$
3. $G(s) = \dfrac{s-1}{s+1}$

Detailed summary of plots on Moodle (and in exercises)

Summary

Summary

The steady-state output of a linear system in response to a sinusoid is a sinusoid of the same frequency with a phase shift of:

- $\angle G(j\omega)$ and a magnitude of $|G(j\omega)|$

Why important?

- Very useful method to experimentally capture the dynamics of a system
- Common control objectives expressed in terms of frequency requirements
- Can determine closed-loop behaviour, from open-loop frequency response

In coming weeks we will use the Bode plot of the open-loop system to:

- Compute key metrics defining the robustness of the system
- Shape the response of closed-loop system by ‘modifying’ the bode plot with a controller